How Children View the Equals Sign

MERLIN BEHR
Northern Illinois University
STANLEY ERLWANGER
Concordia University
EUGENE NICHOLS
Florida State University

Much of the number work presented to 6- and 7-year-old children is in the form of open number sentences. In learning the basic addition facts, for example, children are expected to respond to such sentences as 3 + 4 = ☐. The question of what meaning, if any, symbolic sentences such as these have to children was the focus of the interviews discussed in this article.

The mathematical symbols which children encounter early in the learning of arithmetic are +, =, and such expressions as 3 + 4. The symbol = has several meanings to adults. The most basic meaning is an abstraction of the notion of *sameness.* This is an intuitive notion of equality which arises from experience with equivalent sets of objects. This is the notion of equality which we would hope children would exhibit. A more sophisticated notion of equality, which comes as a result of teaching, is that it is an *equivalence relation*.

In conducting the interviews of concern in this article, we raised questions like the following.

(1) Must addition sentences be of a certain form in order to be considered true by children?
(2) Are the sentences 2 + 4 = ☐ and ☐ = 2 + 4 viewed as having the same meaning?
(3) How do children view sentences such as 3 = 3 and 3 = 5?
(4) How do children understand such sentences as 2 + 3 = 3 + 2?

Some insight into children's ideas about such equality sentences was gained through non-structured individual interviews with children from six to twelve years old.

Sentences of the form, a + b = ☐

We present children with a written sentence of the form $a + b =$ ☐ (e.g., 2 + 4 = ☐). First, questions concerning the meaning of the + symbol are put. From their responses we conclude that they are able to read the symbol and understand that it tells them to add.

Next, when asked to tell the meaning of the = symbol, most of the children's responses are consistent with the understanding represented by one six-year-old, Dot, who says, "When two numbers are added, that's what it [answer] turns out to be."

When questioned about the addends (2 and 4 in 2 + 4 = ☐), the children indicate that they understand these to represent numbers. These six and seven-year-olds accept expressions like 2 + 4 to be meaningful, but this configuration of symbols suggests that something must be done. They do not, for example, generally think to 2 + 4 as being a name for 6.

Seven-year-old Dee says that 2 and 4 are numbers but about 2 + 4 she says, "I don't know, the + means you add them together to a number a number", while another seven-year-old, Mel, explains that 2 + 4 is a number because if "you put them both together, it makes another number". A six-year-old accepts 2 and 4 as numbers but not 2 + 4 "because you put a plus there".

Most of the children are able to judge sentences like 2 + 3 = 5 and 2 + 3 = 7 as true or false. Our observations suggest that when children see a sentence like 3 + 4 = ☐, they perceive it as a stimulus for an answer to be placed in the box. Even in the absence of the = symbol and the box, 2 + 4 serves as a stimulus *to do something.*

Sentences of the form ☐ = a + b

Six-year-old Kay reacts to sentences like ☐ = 4 + 5 in two ways; she says it is backwards and rewrites it as 5 + 4 = ☐, or writes over the = and + with + and =, respectively, so it becomes ☐ + 4 = 5. Although Kay can read and solve some of these types of sentences, she has definite ideas about how they should be written.

Let us look at how some other six-year-olds react to sentences of this form. When Cox is presented with ☐ = 1 + 2, he writes 3 = 1 + 2 and reads the sentence aloud as "2 plus 1 equals 3".

Given ☐ = 3 + 5, Eve counts on her fingers, writes 8 =3 + 5 but reads, "5 plus 3 equals 8." Given 6 = 4 + 1, she changes it to 6 = 4 + 10 saying, "6 and 4 makes 10." Again, when given 3 = 2 + 1, she says, "You should put a 5 here [i.e., at 1]." But when asked to read it, she says, "There… [pause] …yeah, that's … I was reading the wrong way again." Eve explains that 2 + 3 = 5 is easier than 5 = 2 + 3 "because it's [5 =] on this side, and I'm used to having it on that side [= 5]."

Another six-year-old, Tob, reacts to ___ = 2 + 5, and by scribbling over ___ =, and he changes ___ = 2 + 5 to 2 + 5 = ___. He explains that ___ = 2 + 5 is "backwards" and asks the interviewer, "Do you read backwards?"

The seven-year-olds Mel and Dee react as follows. When Mel is given ☐ = 3 + 5, he writes 8 = 3 + 5 and reads, "8 equals 3 plus 5." On the other hand, Dee also writes 8 = 3 + 5, but reads it, "5 plus 3 equals 8."

We observe that only Mel accepts a sentence like ☐ = 2 + 5. The other children resist sentences of this form and change them to another form: 2 + 5 = ☐ or ☐ + 2 = 5.

Sentences presented orally

So far we have considered children's reactions to written sentences. How do they react when these sentences are presented orally?

In the interview, the sentence "5 equals 2 plus 3" is read to six-year-old Kay. She indicates that this is acceptable, saying "Yeah … [and] …'cause … see, like 2 … 3, and here's plus and here's two [writing 2 + 3 =]" and then writes the answer, 5. The interviewer then writes 8 = 3 + 5. To this Kay responds, "that's the wrong answer [changing 8 = 3 + 5 to 8 + 3 =5 by writing + and = over the = and + signs, respectively] … [and] … you couldn't make it 8 equals 3 plus 5 because that means that [8] would be the answer. But it's at the wrong end!"

When six-year-old Eve is presented orally with the statement, "5 is equal to 2 plus 3," she says, "It's not right; that [3] should be a 7." It appears that Eve perceives "5 is equal to 2" as 5 + 2. Neither did Eve accept the orally presented statement "3 is equal to 2 plus 1"; she objects, saying, "… you put a 1 instead of a 5." Again, it appears that she perceives 3 = 2 … as 3 + 2 … . On the other hand, she accepts the orally presented statement, "2 plus 1 equals 3."

When we compare Kay and Eve, we see that Eve displays a rigidity of form in both written and orally presented sentences, while Kay displays this rigidity only in written statements.

"Non-action" sentences

The children's reactions we have considered thus far concern sentences that differ only in the order of the symbols, for example 2 + 3 = 5 and 5 = 2 + 3. These sentences are more frequently encountered by children that are sentences such as 3 = 3, 3 = 5, 2 + 1 = 1 + 2, and 4 + 1 = 2 + 3. Sentences with no plus sign (e.g., 3 = 3), or more than one plus sign (e.g., 2 + 1 = 1 + 2) do not suggest an action. Rather, these sentences require a judgement about their truth-value. How do children react to such sentences?

Six-year-old Kay is given the written sentences 3 = 5 and 3 = 3. When presented with 3 = 5, she says, "Cross that line out [and proceeds to write over the equal sign to change 3 = 5 to 3 + 5]." To 3 = 3 she responds, "Nope … now you could fix that by going like this [changes 3 = 3 to 0 + 3 = 3]."

When Cox is given the written statement 3 = 5, he counts on his fingers and writes 3 = $_8$5. Asked to read what he has written, Cox responds, "5 … there's no plus … [and] … I'll put a plus in the middle." He then changes 3 = $_8$5 to 3 + = $_8$5 and when asked to read he says, I'm going to read backwards, 3 plus 5 equals 8," touching each symbol as he reads it. Responding similarly to 3 = 3 ("It's wrong because there's no plus"), Cox changes 3 = 3 to 3 + = $_6$3.

Eve does not accept 3 = 5 but changes it to 2 + 3 = 5. She explains, "You put a plus there and a 2 there … 'cause then that makes five." Given the written statement 3 = 3, she says, "You should put a plus here and a …" while she changes 3 = 3 to 0 + 3 = 3.

When Tob is presented with the written sentence 3 = 3, he says, "Now if you had a straight line like that [changes 3 = 3 to 3 – 3] it'd be 'subtractly'; it'd be zero [proceeds to complete 3 – 3 to 3 – 3 = 0]." When presented with 3 = 5, he responds similarly, changing 3 = 5 to 3 – 5 and then completes it to 3 – 5 = 0.

These are representative reactions of six-year-old children. These and other interviews with seven-year-old children reveal that when confronted with 3 = 5, the children change it to 3 + 5 = 8, or 3 – 5 = 0. When presented with 3 = 3, they change it to 0 + 3 = 3, or 3 + 3 = 6, or 3 – 3 = 0. That is, each equality statement is transformed into an addition or subtraction sentence.

Thus, six- and seven-year-old children reject sentences of the form a = b. Such sentences are modified to the form a + b = ☐ or a – b = ☐. The "non-action" sentences are modified to "action" sentences with a sum or difference.

It can be argued that the perceptions of equality sentences held by six- and seven-year-old children are an expected outcome of the instruction the children receive. But, what about children eight years old and older? Does the exposure to statements of commutative and associative properties broaden the children's concept of equality?

Let us examine the thinking of eight-year-old May. When asked what 3 = 3 means, May says, "Well, I don't know, but I can guess. It could be the end of some adding or subtracting." When asked if she would like to fix it up, she writes 3 + 0 = 3.

Does children's thinking change by the time they reach age twelve? Tay, twelve years old, when asked about the meaning of 3 = 3, says that this could mean "0 minus 3 equals 3" and writes 0 – 3 = 3. Upon further questioning, she changes 3 = 3 to 6 – 3 = 3, then to 7 – 4 = 3, which she reads, "7 minus 4 equals 3."

Equality statements with two plus signs

Relational statements such as 2 + 3 = 3 + 2 differ from statements such as 3 + 2 = 5 in that they have two plus signs.

Six-year-old Kay is presented with the sentences 2 + 3 = 5 and 3 + 2 = 5. About these she says, while pointing to the 3s and 2s in pairs, "They're the same question." The interviewer then writes 2 + 3 = 3 + 2 and asks Kay if this is all right. "Nope," says Kay, "'cause see, you wrote two of the same thing … [and] … you put 2 plus 3 equals 3 plus 2 … [and] … you didn't do it right." She then changes 2 + 3 = 3 + 2 to 2 + 3 = 5 and 3 + 2 = 5. When presented with 1 + 5 = 5 + 1, she first changes the sentence to 1 + 5 = 1 + 5 and then to 1 + 5 = 1 + 5 = . She indicates that an answer should be written to the right of the second equal sign.

Two other six-year-olds, Cox and Eve, react to statements of the same form as follows. When Cox is presented with the written sentence 1 + 2 = 2 + 1, he changes this to 1 + 2 = 2 + 1. He reads 1 + 2 = 2 + 1 as
3 3 3 3
follows: "1 plus 2 equals 3" [touching the symbols in 1 + 2 = as they are read] and
3
"1 plus 2 equals 3" [touching the symbols in = 2 + 1 as each is read], because it [2 + 3]
3
doesn't have anything … [you] forgot to put the 5." She writes 3 + 2 = 5 and 2 + 3 = 5.

Here are some of the reactions of seven-year-olds. Mel accepts 3 + 2 = 2 + 3 "because they both have the same numbers. Only 2 plus 3 is backwards." Mel notes a difference between 3 + 2 = 2 + 3 and 4 + 1 = 2 + 3 explaining that "Two four [pointing at 2 and 4] one three [pointing at 1 and 3] doesn't rhyme. They are sort of equal because they both equal 5. They don't go together; not made the same … they both equal 5, but they're not the same."

Dee does not accept 1 + 4 = 4 + 1 "because you need a plus there [=]." Similarly, she does not accept 3 + 2 = 2 + 3 "because you need to change it [=] to plus." She changes 3 + 2 = 2 + 3 to 3 + 2 + 2 + 3 = 10.

An essential observation to be made here is that these children do not view sentences like 3 + 2 = 2 + 3 as being sentences about number relationships. They do not see such sentences as indicating the sameness of two sets of objects. Indeed, it appears that the children considered these as "do something" sentences. In most cases the presence of a plus sign along with two numerals suggests that another number, an answer, is to be found. Moreover, at least in the case of Mel, it appears that he is concerned about the "sameness" of two expressions in a symbolic sense. To Mel 3 + 2 = 2 + 3 is O.K. because the two expressions 3 + 2 and 2 + 3 have the same numerals, but 4 + 1 = 2 + 3 is not O.K. because they do not have the same numerals.

The issue which originally motivated this study was whether children consider equality to be an operator or a relation. As an operator symbol, = would be a "do something signal". As a relational symbol, = suggests a comparison of the two members of an equality sentence. These interviews suggest that children consider the symbol = as a "do something signal". There is a strong tendency among all of the children to view the = symbol as being acceptable in a sentence only when one (or more) operation signs (+, –, etc.) precede it. Some children, in fact, tell us that the answer must come after the equal sign. We observe in the children's behaviour an extreme rigidity about written sentences, an insistence that statements be written in a particular form, and a tendency to perform actions (e.g., add) rather than to reflect, make judgements, and infer meaning. Moreover, we have some evidence to suggest that children do not change in their thinking about equality as they get older.

This paper is a shortened version of a paper with the same title available from ERIC, No. ED 144802.